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Topological wave functions and the 4D-5D lift

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Abstract: We revisit the holomorphic anomaly equations satisfied by the topological string amplitude from the perspective of the 4D-5D lift, in the context of "magic" $\mathcal{N} = 2$ supergravity theories. In particular, we interpret the Gopakumar-Vafa relation between 5D black hole degeneracies and the topological string amplitude as the result of a canonical transformation from 4D to 5D charges. Moreover we use the known Bekenstein-Hawking entropy of 5D black holes to constrain the asymptotic behavior of the topological wave function at finite topological coupling but large Kähler classes. In the process, some subtleties in the relation between 5D black hole degeneracies and the topological string amplitude are uncovered, but not resolved. Finally we extend these considerations to the putative one-parameter generalization of the topological string amplitude, and identify the canonical transformation as a Weyl reflection inside the 3D duality group.

Keywords: Black Holes in String Theory, Topological Strings.

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Contents

1. Introduction

BPS black holes in $\mathcal{N} = 2$ supergravity theories have attracted revived attention recently, with the discovery of deep connections between topological strings and the entropy of four-dimensional [1] and five-dimensional [2, 3] black holes, as well as a direct relation between 4D black holes and 5D black holes and black rings, often known as the 4D-5D lift [4 − 7]. These advances have also led to much progress in our understanding of the topological string amplitude, in particular with regard to its wave function character [8 − 14].

In this note, we revisit the holomorphic anomaly equations satisfied by the topological string amplitude from the perspective of the 4D-5D lift. Our analysis is based on the construction in [12], where the standard topological amplitude [15] (BCOV) was recast into a purely holomorphic wave function satisfying a generalized heat equation, as first suggested in [8] (see [13] for a closely related construction). In the same work, the algebraic nature of the topological amplitude was elucidated in the context of so-called "magic" $\mathcal{N} = 2$ supergravity theories [16], characterized by the fact that their moduli space is a symmetric space. Some of these models are known to be consistent quantum $\mathcal{N} = 2$ theories [17], while others arise as truncations of theories with higher supersymmetry (see [18, 19] for recent progress on this issue).

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The outline of this note is as follows. In section 2, we review the relevant results from [12] pertaining to "magic" supergravities. In section 3.1, we observe that the relation between the charges of 4D and 5D black holes related by the 4D-5D lift is a canonical transformation. This motivates us to introduce a new "5D" polarization for the topological wave function, $\Psi_{5D}(Q_i, J)$, related to the standard "real" polarization $\Psi_{\mathbb{R}}(p^I)$ by an appropriate Bogoliubov transformation. In section 3.2, we show that this relation is an instance of the Gopakumar-Vafa connection between 5D black holes and topological strings [2, 3], provided we identify $\Psi_{5D}(Q_i, J)$ with the degeneracies of 5D BPS black holes. In section 3.3, by exploiting the known Bekenstein-Hawking-Wald entropy of 5D black holes, we constrain the asymptotic behavior of the topological string amplitude at finite coupling but for large Kähler classes. In section 4, we extend this 5D polarization to the putative "generalized topological amplitude" introduced in [12], which remains to be constructed, and identify the canonical transformation as a particular Weyl reflection inside the 3D duality group. The details of two computations are relegated in appendices A and B.

2. Magic supergravities and topological wave functions

In this section, we briefly review the main results in [12] on the algebraic nature of the topological string amplitude in "magic" $\mathcal{N} = 2, D = 4$ supergravity theories. Some useful background can be found in [16, 20].

In these models, the vector multiplet moduli space is a Hermitian symmetric tube domain $\mathcal{M} = G/K$ (a very special case of a special Kähler manifold), $G = \mathrm{Conf}(J)$ is the "conformal group" associated to a Jordan algebra J of degree three, K is the maximal compact subgroup of G, a compact real form of the "reduced structure group" $\mathrm{Str}_0(J)$, and the role of the phase space $H^{\mathrm{even}}(X, \mathbb{R})$ in type IIA compactifications on a Calabi-Yau three-fold X (or $H^3(X, \mathbb{R})$ in type IIB compactifications) is played by the "Freudenthal triple" associated to J, namely the real vector space

$$V = \mathbb{R} \oplus J \oplus J \oplus \mathbb{R} \tag{2.1}$$

equipped with the symplectic form

$$\omega = dp^0 \wedge dq_0 + dp^i \wedge dq_i \equiv dp^I \wedge dq_I \tag{2.2}$$

where (p^0, p^i, q_i, q_0) are the coordinates along the respective summands in (2.1). V admits a linear action of G which preserves the symplectic form ω, and leaves the quartic polynomial

$$I_4(p^I, q_I) = 4p^0 N(q_i) - 4q_0 N(p^i) + 4\partial_{p^i} N(p^j)\partial_{q_i} N(q_j) - (p^0 q_0 + p^i q_i)^2 \tag{2.3}$$

invariant. The symplectic space V may be quantized by replacing (p^I, q_I) by operators $\hat{p}^I = p^I, \hat{q}_I = \mathrm{i}\hbar\, \partial/\partial p^I$ acting on the Hilbert space $\mathcal{H}$ of L^2 functions of $n_v + 1$ variables p^I, generating the Heisenberg group H with center $Z = -\mathrm{i}\hbar$,

$$[\hat{p}^I, \hat{q}_J] = Z\, \delta^I_J \; . \tag{2.4}$$

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The linear action of G on V leads to a unitary action of G on $\mathcal{H}$ by generators in the universal enveloping algebra of H[1]

$$S^i \mapsto -\frac{\mathrm{i}}{2}\hbar^2\, C^{ijk}\frac{\partial^2}{\partial p^j\partial p^k} - \hbar\, p^i\frac{\partial}{\partial p^0}, \qquad T_i \mapsto \frac{\mathrm{i}}{2}C_{ijk}p^j p^k - \hbar\, p^0\frac{\partial}{\partial p^i}, \tag{2.5a}$$

$$R_i^j \mapsto -\delta_i^j\, \hbar\, p^0\frac{\partial}{\partial p^0} + \hbar\, p^j\frac{\partial}{\partial p^i} - \frac{1}{2}C_{ikl}\, C^{jnl}\,\hbar\left(p^k\frac{\partial}{\partial p^n} + \frac{\partial}{\partial p^n}p^k\right), \tag{2.5b}$$

$$D \equiv \frac{3}{n_v}R_i^i \mapsto -3\hbar\, p^0\frac{\partial}{\partial p^0} - \hbar\, p^i\frac{\partial}{\partial p^i} - \frac{1}{2}\hbar\,(n_v+3). \tag{2.5c}$$

Here, C_{ijk} is the cubic norm form of J, related to the prepotential F_0 describing the vector multiplet moduli space $\mathcal{M}$ via

$$F_0 = \frac{1}{6}\frac{C_{ijk}X^iX^jX^k}{X^0} \equiv \frac{N(X^i)}{X^0}\,, \tag{2.6}$$

and C^{ijk} is the “adjoint norm form”, satisfying the “adjoint identity”

$$Q_i = \frac{1}{2}C_{ijk}Q^j\,Q^k \quad \Leftrightarrow \quad Q^i = \frac{1}{\sqrt{N(Q_i)}}\,\frac{1}{2}C^{ijk}Q_j\,Q_k\,, \quad N(Q_i) \equiv \frac{1}{6}C^{ijk}Q_iQ_jQ_k\ . \tag{2.7}$$

Thus, the Hilbert space $\mathcal{H}$ furnishes a unitary representation of the “Fourier-Jacobi group” $\tilde{G} = G \ltimes H$, known as the Schrödinger-Weil representation.

Moreover, in [12] (without any assumption of “magicness”), a sequence of transformations was constructed which takes the topological partition function $\Psi_{\mathrm{BCOV}}(t^i,\bar{t}^{\bar{i}};x^i,\lambda)$ from [15], subject to two sets of holomorphic anomaly equations, into a purely holomorphic wave function $\Psi_{\mathrm{hol}}(t^i;y_i,w)$ satisfying a single heat equation[2]

$$\left[\frac{\partial}{\partial t^i} - \frac{\mathrm{i}}{2}C_{ijk}\frac{\partial^2}{\partial y_j\partial y_k} + y_i\frac{\partial}{\partial w}\right]\,\Psi_{\mathrm{hol}}(t^i;y_i,w) = 0\ . \tag{2.8}$$

In magic cases, it was further shown that this holomorphic wave function can be viewed as a matrix element

$$\Psi_{\mathrm{hol}}(t^i;y_i,w) = \langle\Psi|\exp\left(y_i\hat{p}^i + (w - t^iy_i)\hat{p}^0\right)\,\exp(t^iT_i)\,|\Omega_0\rangle \tag{2.9}$$

where $|\Omega\rangle_0$ is the “vacuum” of the Schrödinger-Weil representation, annihilated by $\widehat{q}_I$, S_i and the traceless part of R^i_j, and with charges

$$D\,|\Omega\rangle_0 = -\frac{1}{2}(n_v+3)|\Omega\rangle_0\,, \quad Z\,|\Omega\rangle_0 = -\mathrm{i}|\Omega\rangle_0\ . \tag{2.10}$$

[1]For later convenience, we have flipped the sign of $\hat{q}_I$ and Z with respect to [12], and reinstored a general value for $\hbar$.

[2]This $\Psi_{\mathrm{hol;GNP}}(t^i;y_i,w)$ is related to the holomorphic wave function $\Psi_{\mathrm{hol,ST}}(t^i;\lambda,\epsilon^0,\epsilon^i)$ introduced in [13] by setting $\lambda=1$ using homogeneity, and Fourier transforming (ϵ^0,ϵ^i) into (w,y_i). $\Psi_{\mathrm{hol,ST}}$ arises as the holomorphic limit of the BCOV topological amplitude $\Psi_{\mathrm{BCOV}}(t^i,\bar{t}^{\bar{i}}\to\infty,x^i,\lambda)$, whereas $\Psi_{\mathrm{hol;GNP}}$ may be obtained directly from Ψ_{BCOV} without any limiting or integration procedure.

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The heat equation (2.8) (and ultimately the holomorphic anomaly equations of [15]) can then be shown to follow from the operator identity in the Schrödinger-Weil representation of $\tilde{G}$,

$$Z\,T_i = \hat{p}^0 \hat{q}_i + \frac{1}{2} C_{ijk}\, \hat{p}^j \hat{p}^k \,, \tag{2.11}$$

It is also useful to introduce the operator

$$2\hat{J} = \frac{2}{3} Z\, \hat{p}^i\, T_i + \hat{p}^0 \left(\hat{p}^0 \hat{q}_0 + \frac{1}{3} \hat{p}^i \hat{q}_i \right) = \hat{p}^0 (\hat{p}^0 \hat{q}_0 + \hat{p}^i \hat{q}_i) + \frac{1}{3} C_{ijk}\, \hat{p}^i\, \hat{p}^j\, \hat{p}^k \,, \tag{2.12}$$

whose significance will become apparent shortly.

3. Topological wave functions and black hole entropy

Our starting point is the observation that the right-hand sides of (2.11) and (2.12) formally give the electric charges Q_i and angular momentum J

$$Q_i = p^0\, q_i + \frac{1}{2} C_{ijk}\, p^j\, p^k \,, \tag{3.1a}$$

$$2J = p^0 (p^0 q_0 + p^i q_i) + \frac{1}{3} C_{ijk}\, p^i\, p^j\, p^k \,, \tag{3.1b}$$

of the 5D black hole (or more generally black ring) related to the 4D black hole with charges (p^0, p^i, q_i, q_0) by the 4D-5D lift [4, 6, 7]. Indeed, it is now well-known that a four-dimensional BPS black hole with $D6$ brane charge $p^0 \neq 0$ and arbitrary $D4, D2, D0$ brane charges p^i, q_i, q_0 in type IIA string theory compactified on X may be viewed at strong coupling as a 5D black ring carrying electric M2-brane charges Q_i, M5-brane dipole moments $P^i = -p^i/p^0$ and angular momentum $J_\psi = J$, wound around the circle of a Taub-NUT space with NUT charge p^0 [4, 6, 7]. In the absence of D4-brane charge, the 5D configuration reduces to a single 5D black hole placed at the tip of the Taub-NUT space, as found in [5]. Indeed, with this assignment of charges it may be shown that the Bekenstein-Hawking entropy of the 4D and 5D black holes agree up to the orbifold factor $1/|p^0|$ [5, 6]. In the context of $\mathcal{N} = 2$ magic supergravities, this amounts to the identity

$$S_{4D} = \pi \sqrt{I_4(p^I, q_I)} = \frac{2\pi}{|p^0|} \sqrt{N(Q_i) - J^2} = \frac{1}{|p^0|} S_{5D} \tag{3.2}$$

where I_4 is the quartic invariant in (2.3) [21]. It should also be noted that the charges Q_i, J defined in (3.1) are invariant under the "spectral flow"

$$p^0 \to p^0 \,, \qquad p^i \to p^i + p^0 \ell^i \,, \quad q_i \to q_i - C_{ijk}\, p^j \ell^k - \frac{p^0}{2} C_{ijk} \ell^j \ell^k \,, \tag{3.3a}$$

$$q_0 \to q_0 - \ell^i q_i - \frac{1}{2} C_{ijk}\, p^i \ell^j \ell^k - \frac{p_0}{3} C_{ijk}\, \ell^i \ell^j \ell^k \,. \tag{3.3b}$$

which corresponds to switching on a flux on the Taub-NUT space [5].

3.1 A 5D polarization for the topological amplitude

The form (2.11) of the holomorphic anomaly equations suggests introducing a new polarization where the operators $\hat{Q}_i$ and $\hat{J}$ are diagonalized. For this purpose, we note that, at the classical level, the 5D charges (Q_i, P^i, J), supplemented by an extra charge $p_J = 1/p^0$, are obtained from $(p^I; q_I)$ via a canonical transformation generated by

$$S(p^0, p^i; Q_i, J) = -\frac{N(p^i)}{p^0} + Q_i \frac{p^i}{p^0} - \frac{2J}{p^0} \; . \tag{3.4}$$

Indeed, a straightforward computation making use of the homogeneity of N shows that

$$q_I = \frac{\partial S}{\partial p^I}\,, \quad P^i = -\frac{\partial S}{\partial Q_i} = -\frac{p^i}{p^0}\,, \quad p_J = -\frac{\partial S}{\partial J} = \frac{2}{p^0} \tag{3.5}$$

so that

$$dS = q_I\, dp^I - (P^i\, dQ_i + p_J\, dJ) \; . \tag{3.6}$$

This ensures that the change of variables from $(p^I; q_I)$ to $(Q^i, J; P_i, p_J)$ preserves the Darboux form of ω,

$$\omega = dp^I \wedge dq_I = dQ_i \wedge dP^i + dJ \wedge dp_J \; . \tag{3.7}$$

Quantum mechanically, the wave function $\Psi_{5D}(Q_i, J)$ in the "5D" polarization where $\hat{Q}_i$ and $\hat{J}$ are diagonalized is therefore related to the wave function $\Psi_{\mathbb{R}}(p^I)$ in the "real" polarization [9], where $\hat{p}^I$ acts diagonally, via

$$\Psi_{\mathbb{R}}(p^I) = \int \exp\left(-\frac{\mathrm{i}}{\hbar}\, S(p^I; Q_i, J) \right) \Psi_{5D}(Q_i, J)\, dQ^i\, dJ \; . \tag{3.8}$$

Equivalently,

$$\Psi_{\mathbb{R}}(p^I)\, \exp\left(-\frac{\mathrm{i}}{\hbar} \frac{N(p^I)}{p^0} \right) = \int \exp\left(\frac{\mathrm{i}}{\hbar} \frac{2J}{p^0} - \frac{\mathrm{i}}{\hbar} \frac{p^i}{p^0}\, Q_i \right) \Psi_{5D}(Q_i, J)\, dQ^i\, dJ \tag{3.9}$$

Indeed, one may check that the operators

$$\hat{Q}_i \equiv \mathrm{i}\hbar\, p^0\, \frac{\partial}{\partial p^i} + \frac{1}{2} C_{ijk} p^j p^k\,, \qquad 2\hat{J} = \mathrm{i}\hbar\, (p^0)^2 \frac{\partial}{\partial p^0} + \mathrm{i}\hbar\, p^0\, p^i\, \frac{\partial}{\partial p^i} + 2\, N(p^i) \tag{3.10}$$

acting on the l.h.s. of (3.8) lead to insertions of Q_i and $2J$ in the integral on the r.h.s, respectively. In words, we have found that the wave function in the 5D polarization is obtained by Fourier transforming the wave function in the real polarization with respect to $1/p^0$ and p^i/p^0, after multiplication by the tree-level part $e^{-\frac{\mathrm{i}}{\hbar} N(p^i)/p^0}$.

3.2 5D polarization and 5D black hole degeneracies

In order to interpret the result (3.9), we now recall some facts and conjectures on the relation between the topological string amplitude and various invariants of the Calabi-Yau X.

First, recall that the real polarized topological wave function $\Psi_{\mathbb{R}}(p^I)$ is related to the holomorphic topological wave function via [13]

$$e^{F_{\mathrm{hol}}(t^i, \lambda)} = \left(p^0\right)^{\frac{\chi}{24} - 1}\, \Psi_{\mathbb{R}}(p^I)\,, \quad \lambda = \frac{4\pi}{\mathrm{i}p^0}\,, \quad t^i = \frac{p^i}{p^0} \; . \tag{3.11}$$

where $F_{\rm hol}(t^i,\lambda)$ is the holomorphic limit $\bar{t^{\bar{i}}} \to \infty$ of the topological partition function $F(t^i,\bar{t^{\bar{i}}},\lambda)$.

Second, recall that the Gopakumar-Vafa conjecture [2, 3] relates the indexed partition function of 5D spinning BPS black holes to the topological amplitude,[3]

$$e^{F_{\rm hol}(t^i,\lambda)-F_0(t^i,\lambda)} = \sum_{Q_i,J} \Omega_{\rm 5D}(Q_i,J)\, e^{-2\lambda J+2\pi i Q_i t^i} \ . \tag{3.12}$$

This conjecture also includes a relation to the BPS invariants n_Q^g of the Calabi-Yau X [2],

$$\begin{aligned} e^{F_{\rm hol}(t^i,\lambda)-F_{\rm pol}(t^i,\lambda)} =& [M(e^{-\lambda})]^{-\chi/2} \prod_{Q_i>0,k>0} \left(1-e^{-k\lambda+2\pi {\rm i} Q_i t^i}\right)^{kn_Q^0} \\ &\times \prod_{Q_i>0,g>0} \prod_{\ell=0}^{2g-2} \left(1-e^{-(g-\ell-1)\lambda+2\pi {\rm i} Q_i t^i}\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell} n_Q^g} \end{aligned} \tag{3.13}$$

Here,

$$F_{\rm pol}(t^i,\lambda) = -\frac{(2\pi {\rm i})^3}{\lambda^2} N(t^i) - \frac{2\pi {\rm i}}{24} c_i t^i \tag{3.14}$$

is the "polar part" of $F_{\rm hol}(t^i,\lambda)$, and $M(q) = \prod (1-q^n)^{-n}$ is the Mac-Mahon function. Unfortunately, both the BPS invariants n_Q^g and the 5D black hole degeneracies $\Omega_{\rm 5D}(Q_i,J)$ so far lack a proper mathematical definition. This is in contrast to the now well-established relation between Gromov-Witten and Donaldson-Thomas invariants [23, 24],

$$e^{F_{\rm hol}(t^i,\lambda)-F_{\rm pol}(t^i,\lambda)} = [M(e^{-\lambda})]^{-\chi/2} \sum_{Q_i,J} (-1)^{2J}\, N_{\rm DT}(Q_i,2J)\, e^{-2\lambda J+2\pi i Q_i t^i} \tag{3.15}$$

where $N_{DT}(Q_i,2J)$ are the Donaldson-Thomas invariants. Physically, the latter count the bound states of one D6-brane with $2J$ D0-branes and Q_i D2-branes wrapped along the i-th cycle in $H^{\rm even}(X,\mathbb{R})$.

Finally, in [25], the 4D-5D lift was used to argue that $N_{\rm DT}(Q_i,2J) \sim \Omega_{\rm 5D}(Q_i,J)$, thereby giving a heuristic derivation of the Gopakumar-Vafa conjecture (3.12). However, this argument does not account for the powers of the Mac-Mahon function in (3.15) relative to (3.12), nor for the sign $(-1)^{2J}$. There is also a discrepancy (most likely due to a difference in the treatment of the center of motion degrees of freedom) between the prediction of the infinite product representation (3.13), $N_{\rm DT}(Q,2J) = \sum_g \binom{2g-2}{2J+g-1} n_Q^g$, and the considerations in [3, 26], which lead to $\Omega_{5D}(Q,J) = \sum_g \binom{2g+2}{2J+g+1} n_Q^g$. Without attempting to resolve these issues, we shall regard (3.12) as a definition of the 5D black hole degeneracies $\Omega_{\rm 5D}(Q_i,J)$, and later assume that $\log \Omega_{\rm 5D}(Q_i,J)$ is given by the Bekenstein-Hawking-Wald formula, barring any "miraculous" cancellations.

Substituting (3.11) into (3.12) and setting $c_i=0$ for simplicity, we obtain

$$(p^0)^{\frac{\chi}{24}-1}\, \Psi_{\mathbb{R}}(p^I)\, \exp\left(\frac{{\rm i}\pi}{2}\frac{N(p^I)}{p^0}\right) = \sum_{Q_i,J} \exp\left(8\pi {\rm i}\frac{J}{p^0} + 2\pi {\rm i} Q_i \frac{p^i}{p^0}\right) \Omega_{5D}(Q_i,J) \ . \tag{3.16}$$

[3]Here and below, we follow the conventions in [22], up to minor changes of notation $g_{\rm top} \to \lambda, n \to 2J, \beta_i \to Q_i$.

Barring the power of p^0, allowing for rescalings of Q_i and J and setting the Planck constant to $\hbar = -2/\pi$, we see that (3.9) and (3.16) are consistent provided the topological wave function in the 5D polarization has delta function support on integer charges Q_i, J, with weights equal to the 5D black hole degeneracies,[4]

$$\Psi_{5D}(Q_i, J) \sim \sum_{Q'_i, J'} \Omega_{5D}(Q'_i, J')\, \delta\Big(Q'_i - \frac{1}{4} Q_i, J' + \frac{1}{8} J \Big)\,. \tag{3.17}$$

The power of p^0 in (3.16) may be attributed to a quantum ordering ambiguity invisible in the semi-classical discussion in the previous Subsection, or may be absorbed in a redefinition of $\Omega_{5D}(Q_i, J)$. Note also that (3.9) was motivated in the context of magic supergravities, but that (3.16) holds (to the same extent as (3.12)) in arbitrary $\mathcal{N} = 2$ string compactifications.

Thus, we conclude that the 5D black hole degeneracies $\Omega_{5D}(Q_i, J)$ can be viewed as a wave function in a particular "5D" polarization, related to the standard real polarization by the intertwining operator (3.16). The fact that the degeneracies $\Omega_{5D}(Q_i, J)$ can be interpreted as components of a wave function in a representation space of the group $\tilde{G}$ gives some support to the general expectation (voiced e.g. in [21, 27]) that they should arise as Fourier coefficients of a certain automorphic form of $\tilde{G}$.

3.3 Black hole entropy and asymptotics of the topological amplitude

Assuming the validity of the Gopakumar-Vafa conjecture (3.12) (and regardless of the correctness of the identification (3.17)), we can use our knowledge of the entropy of 5D black holes to constrain the asymptotic behavior of the topological string amplitude. Recall that the Bekenstein-Hawking entropy of 5D BPS spinning black holes is given at tree level by [28]

$$S_{5D} = 2\pi \sqrt{Q^3 - J^2} \tag{3.18}$$

where Q is to be expressed in terms of the electric charges via

$$Q^{3/2} = \frac{1}{6} C_{ijk} Q^i Q^j Q^k\,, \quad Q_i = \frac{1}{2} C_{ijk} Q^j Q^k \tag{3.19}$$

Equation (3.18) is valid in the limit where Q_i and J are scaled to infinity, keeping the ratio J^2/Q^3 fixed and less than unity. Taking into account higher-derivative corrections of the form $\int c_i\, A^i \wedge R \wedge R$ together with their supersymmetric partners, the Bekenstein-Hawking-Wald entropy becomes [29]

$$S_{5D} = 2\pi \sqrt{Q^3 - J^2} \left(1 + \frac{c_i\, Q^i}{16} \frac{Q^{3/2}}{Q^3 - J^2} + \mathcal{O}(c^2) \right) \tag{3.20}$$

which is valid in the same regime. The free energy of rotating BPS black holes in 5 dimensions in a thermodynamical ensemble with electric potentials ϕ^i and angular velocity ω,

$$F_{5D}(\phi^i, \omega) \equiv \mathrm{Extr}_{Q_i, J} \left[S_{5D} - \omega J - Q_i \phi^i \right] \tag{3.21}$$

[4]The factors of $1/4$ and $-1/8$ in this equation are convention-dependent, and so is the value of $\hbar$.

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is easily computed to first order in c_i, (see appendix B for details)

$$F_{5D}(\phi^i,\omega) = -\frac{1}{\pi^2}\,\frac{N(\phi^i)}{1+(\omega/2\pi)^2} - \frac{1}{8}c_i\phi^i + \mathcal{O}(c^2)\ . \tag{3.22}$$

This results holds for arbitrary, non-magic supergravities in 5D, and is considerably more elegant than its Legendre dual (3.20).[5]

The free energy (3.22) provides the classical (saddle point) approximation to the integral in (3.9). The fluctuation determinant around the saddle point may be computed in the magic cases using the results in [21]. Setting

$$\phi^i = -2\pi\mathrm{i}\, t^i\,, \quad \omega = 2\lambda\,, \tag{3.23}$$

we find

$$\Psi_{\mathbb{R}}(X^I) \sim \lambda^{\frac{\chi}{24}-1}\left(\frac{N(t_i)}{\lambda^2+\pi^2}\right)^{\frac{n_v+3}{6}} \exp\left[-(2\pi i)^3\left(\frac{N(t^i)}{\lambda^2} - \frac{N(t^i)}{\pi^2+\lambda^2}\right) + \frac{2\pi i}{8}c_i t^i + \dots\right] \tag{3.24}$$

where the prefactor can be trusted in magic cases only. In the scaling limit where the Bekenstein-Hawking formula can be trusted, the topological coupling λ at the saddle point is fixed while t^i are scaled to infinity, so that the terms displayed in (3.22) are the first two in a systematic expansion at large t^i, for fixed λ. It is noteworthy that the terms proportional to $1/\lambda^2$ and $1/(\pi^2+\lambda^2)$ in the exponent cancel in the limit of large λ, leaving a term of order $1/\lambda^4$ only. Incidentally, we note that the linear term in t^i in the exponent induces a correction $Q_i \to Q_i - \frac{1}{8}c_i$ to the 4D-5D lift formulae (3.1), consistent with [30, 29] in the absence of angular momentum, but giving a different correction than the one found in [29] when $J \neq 0$.

On the other hand, at small topological coupling and finite values of t^i, (3.11) yields

$$\Psi_{\mathbb{R}}(p^I) \sim (p^0)^{1-\frac{\chi}{24}} \exp\left[-\frac{\mathrm{i}\pi}{2}F_0(p^I)\right] \sim \lambda^{\frac{\chi}{24}-1} \exp\left[-(2\pi\mathrm{i})^3\frac{N(t^i)}{\lambda^2} - \frac{2\pi\mathrm{i}}{24}c_i t^i + \dots\right] \tag{3.25}$$

The semi-classical limit $\lambda \to 0$ at fixed t^i is consistent with the entropy of 4D BPS black holes, a fact which lies at the basis of the OSV conjecture [1]. For completeness, we show in appendix B how (3.25) is consistent with the usual form of the BCOV topological amplitude,

$$\Psi_{\mathrm{BCOV}}(t_i,\bar{t}_i,x^i,\lambda) \sim \lambda^{\frac{\chi}{24}-1} \exp\left(-(2\pi\mathrm{i})^3\frac{C_{ijk}x^i x^j x^k}{\lambda^2} + \mathcal{O}(\lambda^0)\right) \tag{3.26}$$

after performing the sequence of transformations given in [12].

The regimes of validity of (3.24) and (3.25) in principle overlap when λ goes to zero and t^i to infinity. While the two results agree in the strict classical limit, the prefactors do not.

[5] The apparent discrepancy at order c_i with the free energy given in eq. (3.23) of [29] is due to the fact that the chemical potentials e_I are conjugate to the 4D charges rather than the ones measured at infinity in 5D, as recognized in [29]. At zeroth order in c_i, the result (3.22) was known to the second author, R. Dijkgraaf and E. Verlinde in 2005.

Moreover matching the terms in the exponent would require $F_1(t^i) \sim \frac{2\pi i}{8} c_i t^i + \frac{(2\pi i)^3}{\pi^2} N(t^i)$, which violates the assumption that F_1 is grows linearly at large t^i. This discrepancy suggests that the two limits $t^i \to \infty$ and λ do not commute. It would be interesting to understand the physical origin of this phenomenon.

4. The generalized topological amplitude in the 5D polarization

In the previous section, we have shown that the 4D-5D lift formula and Gopakumar-Vafa conjectures had a simple interpretation as a change of polarization in the Schrödinger-Weil representation of the Fourier-Jacobi group $\tilde{G} = G \ltimes H$, where G is the four-dimensional U-duality group and H is the Heisenberg algebra of electric, magnetic and NUT charges.

4.1 Dimensional reduction and extended topological amplitude

Physically, the Fourier-Jacobi group $\tilde{G}$ naturally arises as a subgroup of a larger group G', the duality group after reducing the 4D supergravity (or compactifying) down to 3 dimensions. After dualizing the one-forms into scalars, the theory in 3 dimensions can be expressed as a non-linear sigma model on a quaternionic-Kähler space $\mathcal{M}_3 = G'/\mathrm{SU}(2) \times G_c$, where G_c is a compact form of the duality group G in 4 dimensions [31]. The reduction from G' to its subgroup $\tilde{G}$ corresponds to decoupling gravity in 3 dimensions. In the language of Jordan algebras, $G' = \mathrm{QConf}(J)$ is the "quasi-conformal group" associated to the Jordan algebra J [32, 33, 20]. Its Lie algebra can be obtained by supplementing the solvable group $\tilde{G}$ with the negative roots $\widehat{p}^{I'}, \widehat{q}_I, Z'$, obeying the "dual Heisenberg algebra" $[\widehat{p}^{I'}, \widehat{q}_J'] = Z' \delta^I_J$, and introducing a new Cartan generator $\Delta \equiv [Z, Z']$, such that Z, Z', Δ forms a $Sl(2,\mathbb{R})$ subalgebra commuting with G (see figure 1).

Moreover, the group G' admits a distinguished unitary representation known as the "minimal" representation, whose functional dimension $n_v + 2$ is the smallest among the unitary irreducible representations of G'. The minimal representation of G' extends the Schrödinger-Weil representation of $\tilde{G}$ in the following way: classically, the Freudenthal triple (2.1) is extended into

$$V' = \mathbb{R}_y \oplus \mathbb{R}_{p^0} \oplus J_{p^i} \oplus J_{q_i} \oplus \mathbb{R}_{q_0} \oplus \mathbb{R}_{p_y} \tag{4.1}$$

equipped with the symplectic form

$$\omega' = dy \wedge dp_y + dp^0 \wedge dq_0 + dp^i \wedge dq_i \ . \tag{4.2}$$

The linear space V' turns out to be symplectically isomorphic to the minimal co-adjoint orbit of the complexification of G' (itself isomorphic to the hyperkähler cone over the quaternionic-Kähler space $G'/(\mathrm{SU}(2)\times G_c)$), and therefore admits a holomorphic symplectic action of G' on (4.1). The minimal representation is obtained by quantizing this action (see e.g. [33] for details on this procedure). Quantum mechanically, the minimal representation of G' may be obtained from the Schrödinger-Weil representation (2.5) by allowing the center $Z = i\hbar$ to become dynamical, i.e. supplement the Hilbert space $\mathcal{H}$ of L^2 functions

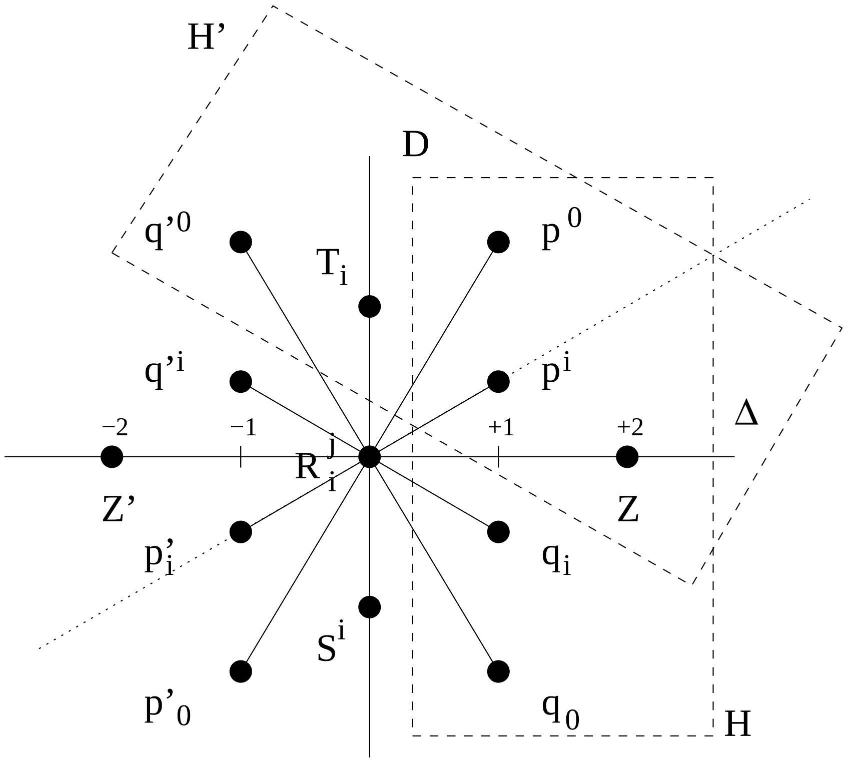

Figure 1: Two-dimensional projection of the root diagram of G', with respect to the split torus (Δ, D). The subgroup G consists of the roots along the vertical axis. The Fourier-Jacobi subgroup $\tilde{G} = G \ltimes H$ consists of the roots on and to right of the vertical axis, together with the Cartan generator Δ. The Heisenberg algebras H and H' are exchanged by a Weyl reflection W with respect to the dotted axis.

of $n_v + 1$ variables p^I with an extra variable y, and set,[6]

$$Z \mapsto \mathrm{i}y^2, \tag{4.3a}$$

$$\mathrm{i}\widehat{q}_0 \mapsto y\frac{\partial}{\partial p^0}, \quad \mathrm{i}\widehat{q}_i \mapsto y\frac{\partial}{\partial p^i}, \quad \mathrm{i}\widehat{p}^i \mapsto \mathrm{i}yp^i, \quad \mathrm{i}\widehat{p}^0 \mapsto \mathrm{i}yp^0, \tag{4.3b}$$

while keeping the same formulae for the action of G as in (2.5c). The rest of the generators of G' are obtained by commuting the generators above with

$$Z' \mapsto \frac{1}{2}\frac{\partial^2}{\partial y^2} - \frac{1}{4y^6}\left(I_4(\widehat{p}^I, \widehat{q}_I) + \kappa\right), \quad \Delta \mapsto y\partial_y + \frac{1}{2} \tag{4.3c}$$

where the constant κ depends on the ordering chosen in $I_4(\hat{p}^i, \hat{q}_i)$. In particular,

$$\mathrm{i}\widehat{q}'_I \equiv [\widehat{q}_I, Z'] \mapsto \mathrm{i}\frac{\partial}{\partial p^I}\partial_y + \frac{1}{y^4}\frac{\partial I_4(\hat{p}^I, \hat{q}_I)}{\partial \hat{p}^I}, \tag{4.3d}$$

$$\mathrm{i}\widehat{p}'^I \equiv [\widehat{p}^I, Z'] \mapsto \mathrm{i}p^I\partial_y - \frac{1}{y^4}\frac{\partial I_4(\hat{p}^I, \hat{q}_I)}{\partial \hat{q}_I}. \tag{4.3e}$$

[6]With this notation the scalar p^I differs from the eigenvalue of $\widehat{p}^I$ by a power of y. We hope that this will not cause any confusion.

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These formulae define the minimal representation in the real polarization, where the operators $\widehat{p}^I$ and Z are diagonalized. At fixed value of y, the representation of the subgroup $\tilde{G} \subset G'$ reduces to the Schrödinger-Weil representation studied in the previous section, after appropriate y-rescalings.

As argued in [12], the relation between $\tilde{G}$ and G' on the one hand, and between the Schrödinger-Weil representation of $\tilde{G}$ and the minimal representation of G' on the other hand, is closely analogous to the relation of the Fourier-Jacobi group $Sl(2,\mathbb{R}) \ltimes H_3$ and Siegel's genus 2 modular group $\mathrm{Sp}(4,\mathbb{R})$, familiar from the mathematical theory of Jacobi and Siegel modular forms [34] (Here H_3 is the three-dimensional Heisenberg algebra $[p,q]=Z$, where (p,q) transform as a doublet of $Sl(2,\mathbb{R})$). In that case, the Schrödinger-Weil representation of $Sl(2) \ltimes H_3$ on L^2-functions of one variable is then given by the restriction of the metaplectic representation of $\mathrm{Sp}(4,\mathbb{R})$ on L^2-functions of two variables, at a fixed value of the center Z. At the automorphic level, the m-th Fourier coefficient of a Siegel modular form with respect to the action of the center Z yields a Jacobi form of $Sl(2,\mathbb{Z}) \times H_3$ of index m [34]. Based on this analogy, it was suggested in [12] that, in cases where the vector multiplet moduli space is symmetric, the standard BCOV topological amplitude should arise as a Fourier coefficient at $Z=-\mathrm{i}$ of an automorphic form under the larger group $G' = \mathrm{QConf}(J)$, referred to as the "extended topological amplitude". It was further speculated in [12] that the Fourier coefficients at other values of Z yield non-Abelian generalizations of the Donaldson-Thomas invariants.

At this point, we note that the dimensional reduction to 3 dimensions, which has been of great utility in describing four-dimensional stationary black holes [31, 27, 35], is also very useful in order to describe five-dimensional black holes with a U(1) isometry [36 – 39]. The two reductions differ, however, since 5D black holes are best described by reducing the 5D Lagrangian along the time-like direction t first, and then along a space-like direction ψ, while 4D black holes are more conveniently described by first reducing from 5D to 4D along the space-like direction ψ, and then from 4D to 3D along the time-like direction t. The two procedures are related by a Weyl reflection W inside the diffeomorphism group of the (t,ψ) torus, which happens to be the $Sl(2)$ subgroup of G' generated by $\widehat{q}_0, \widehat{q}_0'$ and their commutator [39]. The Weyl reflection W maps the Heisenberg algebra $\{p^I, q_I, Z\}$ (enclosed in the vertical box of figure 1) to the Heisenberg algebra $H' = \{\widehat{q}_0', T_i, \widehat{p}^i, Z, \widehat{p}^0\}$ (enclosed by the tilted box). In particular, the D2 and D0 brane charges $\widehat{q}_i$ and $\widehat{q}_0$ are mapped to T_i and $\widehat{q}_0'$. According to (2.11) and (4.3d) above, the corresponding generators in the minimal representation are given by

$$\mathrm{i}\, T_i = \frac{1}{y^2}\left(\widehat{p}^0 \widehat{q}_i + \frac{1}{2} C_{ijk} \widehat{p}^j \widehat{p}^k\right), \quad \widehat{q}_0' = \frac{1}{y^4}\left[\widehat{p}^0(\widehat{p}^0 \widehat{q}_0 + \widehat{p}^i \widehat{q}_i) + 2N(\widehat{p})\right] + \frac{1}{2y}\widehat{p}^0 \widehat{p}_y \tag{4.4}$$

where $\widehat{p}_y = \mathrm{i}\partial_y$. This are indeed the 5D electric charges Q_i and angular momentum J in (3.1), up to a normalization factor and and additive term in $\widehat{q}_0'$.[7] Moreover, the unit D6-brane charge requirement $p^0 = 1$, appropriate for lifting a 4D black hole to a smooth 5D

[7]Despite the fact that equations (4.4) hold only in the minimal representation, whose semi-classical limit pertains to special solutions whose Noether charge is nilpotent of degree 2, the equality of the conserved charges $(T_i, \widehat{q}_0')$ with the electric charge and angular momentum holds in general [39] .

black hole, is mapped to $Z = -\mathrm{i}$, which is the necessary requirement for the ψ circle bundle over S^2 to be topologically S^3 [39]. Conversely, the vanishing of the time-like NUT charge $Z = 0$ for 4D black holes is mapped to the absence of p^0 charge for 5D black holes [39].

4.2 A 5D polarization for the minimal representation

We now construct the analogue of the 5D polarization in this generalized setting. For this purpose, we need to supplement the 5D charges (Q_i, J) and their canonical conjugate (P^i, p_J) with an extra canonical pair (L, p_L), preserving the fact that (Q_i, J) are related to (p^I, y, q_I, p_y) by (4.4). The canonical transformation generated by

$$S'(p^0, p^i, p_y; Q_i, J, L) = \frac{N(p^i)}{p^0} - \frac{p^i}{p^0} Q_i + \frac{2JL}{(p^0)^2} + L\frac{p_y}{p^0} \tag{4.5}$$

satisfies these conditions (compare to (3.4)). Indeed, after some algebra, one finds that the 5D phase space variables $(Q_i, J, L; P^i, p_J, p_L)$ are related to the 4D phase space variables $(p^I, p_y; q_I, y)$ via

$$Q_i = p^0 q_i + \frac{1}{2} C_{ijk} p^j p^k , \tag{4.6a}$$

$$2J = \frac{1}{y}\left[p^0(p^0 q_0 + p^i q_i) + \frac{1}{3} C_{ijk} p^i p^j p^k\right] + \frac{1}{2} p^0 p_y \tag{4.6b}$$

$$L = p^0 y , \quad P^i = \frac{p^i}{p^0} , \quad p_J = \frac{y}{p^0} , \tag{4.6c}$$

$$p_L = \frac{1}{y\,(p^0)^2}\left[p^0(p^0 q_0 + p^i q_i) + \frac{1}{3} C_{ijk} p^i p^j p^k\right] - \frac{p_y}{2p^0} \tag{4.6d}$$

The generating function of the canonical transformation from (p^0, p^i, y) to (Q_i, J, L) is obtained by Legendre transforming (4.5) with respect to p_y, which removes the last term in (4.5) and sets $y = L/p^0$ consistently with (4.6c) above. Quantum mechanically, the wave function in the generalized 5D polarization $\Psi_{5D}(Q^i, J, L)$ is therefore related to the generalized wave function in the real polarization $\Psi_{\mathrm{gen}}(p^0, p^i, y)$ via

$$\Psi_{\mathrm{gen}}(p^I, y)\, e^{-\mathrm{i}\, N(p^i)/p^0} = \int \exp\left(2\mathrm{i}\frac{y\,J}{p^0} - \mathrm{i}\,\frac{p^i}{p^0}\,Q_i\right) \Psi_{5D}(Q_i, J, L)\, \delta(L - p^0 y)\, dQ^i\, dJ\, dL\, . \tag{4.7}$$

In this new polarization, the "tilted" Heisenberg algebra H', with center $\widehat{p}^0$ is now canonically represented,

$$\begin{aligned} \widehat{q}'_0 &= 2\mathrm{i}J , & T_i &= \mathrm{i}\, Q_i , & & \\ Z' &= \frac{1}{2} L\partial_J , & \widehat{p}^i &= L\partial_{Q_i} , & \widehat{p}^0 &= \mathrm{i}L\, . \end{aligned} \tag{4.8}$$

In fact, the intertwiner (4.7) represents the action of the Weyl reflection W, which takes H into H'. Thus, all generators in the 5D polarization can be obtained from those in the 4D polarization by reflecting the root diagram in figure 1 along the dotted axis and changing variables

$$2J \to p^0 , \quad Q_i \to y p^i , \quad L \to y^2\, . \tag{4.9}$$

For example,

$$Z = \frac{1}{2} L \partial_J \,, \quad \Delta = L\partial_L - J\partial_J \,, \quad Z' = 2J\partial_L + \frac{2J}{L} - \frac{N(Q_i)}{L^2} \,. \tag{4.10}$$

These results agree and generalize the ones obtained for $G' = G_{2(2)}$ in section 3.7.2 of [33], after performing an overall Fourier transform over all p_I. Note that (4.10) implies that $(2J, L)$ transform linearly as a doublet under the $Sl(2,\mathbb{R})$ symmetry generated by Z, Z', Δ, which is Ehlers' symmetry in four dimensions. Thus, the 5D polarization constructed here would be the most convenient starting point to implement Ehlers' symmetry on the generalized topological string amplitude.

5. Discussion

In this note, motivated by the formal analogy between the holomorphic anomaly equations and the 4D-5D lift formulae for "magic" supergravities, we gave a quantum mechanical interpretation of the Gopakumar-Vafa relation as a Bogoliubov transformation from the real polarization, where the 4D magnetic charge operators $\widehat{p}^I$ operators act diagonally, to the the "5D" polarization, appropriate to the operators $\hat{Q}_i$ and $\hat{J}$. Moreover, we used to the known Bekenstein-Hawking-Wald entropy of 5D BPS black holes to constrain the asymptotic behavior of the topological wave function in the real polarization, at finite topological coupling but large Kähler (or, in the B-model, complex structure) moduli t^i.

In the process we found two relatively minor discrepancies: (i) a yet unexplained shift of genus $g \to g-2$ in the relation between $N_{\rm DT}(Q_i, 2J)$ and $\Omega_{\rm 5D}(Q_i, J)$, and (ii) a disagreement at subleading order in the expected overlapping regime of validity of the asymptotic expansions afforded by the 4D and 5D black hole entropy. The former may probably be solved by a proper accounting of the zero-modes of a 5D black hole at the tip of Taub-NUT space, while the latter suggests a non-commutativity of the limits $\lambda \to 0$ and $t^i \to \infty$. It would certainly be useful to resolve these puzzles, and improve our understanding of 5D black hole micro-states.

In the last section of this paper, we extended the construction of the 5D polarization to the case where gravity is no longer decoupled, and the duality group is enlarged from $G \ltimes H$ to a semi-simple Lie group G'. In particular, we found that the intertwiner from the real to the 5D polarization represents a particular Weyl reflection in the 3D duality group G', which exchanges the two directions in the internal 2-torus. Assuming that a "generalized topological amplitude" living in the minimal representation of G' can really be defined, it is interesting to ask what information it may capture. In [12], it was suggested that $\Psi_{\rm gen}(p^I, y)$ would give access to non-Abelian Donaldson-Thomas invariants of rank y^2. The 5D polarized wave function $\Psi_{5D}(Q_i, J, L)$ constructed herein naturally suggests an interpretation in terms of counting 5D black hole micro-states of charge Q_i, angular momentum J and dipole charge $p^0 \propto L$. It would be interesting to see if this conjecture can be borne out.

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A. Free energy of 5D spinning black holes

In this appendix, we derive eq. (3.22) for the free energy of 5D spinning black holes. We start by computing the Legendre transform of the tree-level entropy (3.18), and incorporate the higher-derivative corrections at the end.

Extremizing (3.21) over J, we find that the extremum is reached at

$$J = -\frac{\omega}{\sqrt{\omega^2+4\pi^2}}\,Q^{3/2} \tag{A.1}$$

leaving

$$F_{5D}(\phi^i,\omega) = \langle \sqrt{\omega^2+4\pi^2}\,Q^{3/2} - Q_i\phi^i \,\rangle_{Q_i}\ . \tag{A.2}$$

The extremum over Q_i is therefore reached at

$$Q^i = \frac{1}{\pi}\frac{\phi^i}{\sqrt{1+(\omega/2\pi)^2}}\,, \tag{A.3}$$

at which point

$$F_{5D}(\phi^i,\omega) = -\frac{1}{\pi^2}\frac{N(\phi^i)}{1+(\omega/2\pi)^2} \tag{A.4}$$

To incorporate the effect of the higher-derivative correction in (3.20), we note that the variation of the tree-level entropy (3.18) with respect to q_i is given, to leading order, by

$$\delta S_{5D} = \frac{\pi Q^{3/2}}{\sqrt{Q^3-J^2}} Q^i \delta Q_i \tag{A.5}$$

where we used the fact that $\delta Q^{3/2} = \frac{1}{2}Q^i\delta Q_i$. Thus, the subleading term in (3.20) is reproduced by setting $\delta Q_i = \frac{1}{8}c_i$. After Legendre transform, the corrected free energy is therefore

$$F_{5D}(\phi^i,\omega) = -\frac{1}{\pi^2}\frac{N(\phi^i)}{1+(\omega/2\pi)^2} - \frac{1}{8}\phi^i c_i + \dots \tag{A.6}$$

Upon scaling Q_i and J to infinity keeping $J/Q^{3/2}$ fixed and less than one, it is easy to see that ω is fixed while ϕ^i go to infinity. The limit $\omega \to \infty$ (corresponding to strong topological coupling) corresponds to black holes near the Kerr bound $J = Q^{3/2}$.

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B. From BCOV to real polarization

In this appendix, we provide a check on (3.11) in the case of "magic" $\mathcal{N}=2$ supergravities, which illuminates the relation between the constructions in [12] and [13]. For this purpose, we postulate the form

$$\Psi_{\mathbb{R}}(p^I) \sim (p^0)^{\alpha} \exp\left[-\frac{\mathrm{i}\pi}{2}\frac{N(p^i)}{p^0}\right] , \tag{B.1}$$

consistent with (3.11) for $\alpha = 1 - \frac{\chi}{24}$, and show that it leads to a BCOV topological partition function of the expected form (3.26) after applying the chain of transformations in [12]. The first step is to obtain the holomorphic wave function via [12]

$$\Psi_{\rm hol}(t^i; w, y_i) = \int dp^I \exp\left(\frac{\mathrm{i}\pi}{4} p^I \tau_{IJ}(X) p^J + \frac{\pi}{2} p^I y_I\right) \psi_{\mathbb{R}}(p^I) . \tag{B.2}$$

where $t^i = X^i/X^0$ and $w = y_0 + t^i y_i$. To evaluate this integral in the saddle point approximation, define $\tilde{p}^i = p^i - p^0 X^i/X^0, \tilde{p}^0 = p^0$. Taylor expanding the r.h.s. at small p^0, it is easy to check that

$$-\frac{1}{2} p^I \tau_{IJ}(X) p^J = \frac{N(\tilde{p}^i)}{\tilde{p}^0} - \frac{N(p^i)}{p^0}) . \tag{B.3}$$

Moreover, defining $\tilde{y}_0 = y_0 + y_i X^i/X^0, \tilde{y}_i = y_i$, we have

$$p^I y_I = \tilde{p}^0 \tilde{y}_0 + \tilde{p}^i y_i . \tag{B.4}$$

Inserting (B.1) into (B.2) and changing variables from p^I to $\tilde{p}^I$ leads then to

$$\Psi_{\rm hol}(X^I, y_I) = \int d\tilde{p}^I \, (\tilde{p}^0)^{\alpha} \exp\left[-\frac{\mathrm{i}\pi}{2}\frac{N(\tilde{p}^i)}{\tilde{p}^0} + \frac{\pi}{2}\tilde{p}^I \tilde{y}_I\right] . \tag{B.5}$$

In the saddle point approximation, using the results in [21], we conclude that

$$\Psi_{\rm hol}(X^I, y_I) \sim (\tilde{y}_0)^{\alpha'} \, [N(\tilde{y}_i)]^{\beta'} \exp\left[\frac{\mathrm{i}\pi}{2}\frac{N(\tilde{y}_i)}{\tilde{y}_0}\right] , \tag{B.6}$$

where (except in the D_n case)

$$\alpha' = -2\alpha - \frac{1}{2}(n_v + 3) , \quad \beta' = \alpha + \frac{1}{6}(n_v + 3) . \tag{B.7}$$

Next, we take the complex conjugate $\Psi_{\rm ahol}(\bar{X}^I, \bar{y}_I)$ of (B.6) and change variable from $\bar{y}_I$ to x^i, λ using

$$x^I \equiv [{\rm Im}\tau]^{IJ} \bar{y}_J = 2\, e^{-\frac{1}{4}\pi \mathrm{i}} \, \lambda^{-1} \, (X^I + x^i \, D_i X^I) . \tag{B.8}$$

The BCOV topological partition function is finally obtained as

$$\Psi_{\rm BCOV}(t^i, \bar{t}^{\bar{i}}, x^i, \lambda) = e^{-f_1(t)} \sqrt{\det[{\rm Im}\tau]} \exp\left(-\pi \, x^I [{\rm Im}\tau]_{IJ} x^J\right) \Psi_{\rm ahol}(\bar{X}^I, \bar{y}_I) \tag{B.9}$$

For magic supergravities, and in the gauge $X^0 = \bar{X}^0 = 1$, equations (B.8) are solved by

$$\bar{\tilde{y}}_0 = \mathrm{i}\, \lambda^{-1} \, e^{-K} , \quad \bar{y}_{\bar{i}} = -\mathrm{i}\, \lambda^{-1} e^{-K} g_{\bar{i}j} \left(x^j - (t^j - \bar{t}^j)\right) , \tag{B.10}$$

Moreover,

$$\det[\mathrm{Im}\tau] = e^{-\frac{n_v+3}{3}K}\,, \quad N(\bar{y}_{\bar{i}}) = \mathrm{i}\lambda^{-3}\, e^{-K}\, N\left(x^j - (t^j - \bar{t}^j)\right)\ . \tag{B.11}$$

Altogether, (B.9) evaluates to

$$\Psi_{\mathrm{BCOV}}(t_i, \bar{t}_i, x^i, \lambda) \sim \lambda^{-\alpha}\left(\frac{N(t^i - \bar{t}^i)}{N\left(x^i - (t^i - \bar{t}^i)\right)}\right)^{\beta'} \exp\left[\frac{i\pi}{2}\frac{N\left(x^i - (t^i - \bar{t}^i)\right)}{\lambda^2} - \pi x^I [\mathrm{Im}\tau]_{IJ} x^J\right] \tag{B.12}$$

The quadratic correction in the exponent cancels the terms of order 0,1,2 in x^i, leaving only the cubic term in the exponent,

$$\Psi_{\mathrm{BCOV}}(t_i, \bar{t}_i, x^i, \lambda) \sim \lambda^{-\alpha}\left(\frac{N(t^i - \bar{t}^i)}{N\left(x^i - (t^i - \bar{t}^i)\right)}\right)^{\beta'} \exp\left(\frac{\mathrm{i}\pi}{2}\frac{N(x^i)}{\lambda^2}\right) \tag{B.13}$$

Thus, we find agreement with the expected form (3.26) provided we set

$$f_1(t) = 0\,, \quad \alpha = 1 - \frac{\chi}{24} \tag{B.14}$$

This provides an independent check on (3.11), which was arrived at in [13] by a rather different line of reasoning from [12]. In particular, the fact that the power of λ in (B.13) turns out to be opposite to the power of p^0 in (B.1) is rather non-trivial.

We note that the second factor in (B.13) contributes to genus one 1-point functions, unless $\alpha = -(n_v + 3)/6$. Although such contributions are perfectly admissible, it is worth noting that the special value of α where they disappear is also the one where (B.1) is invariant under Fourier transform with respect to all p^I [21].

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